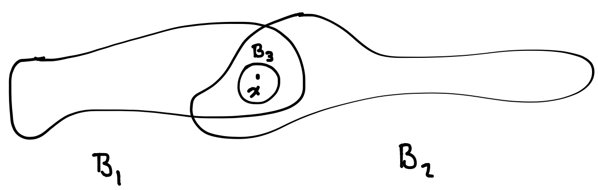
## Bases

With metric spaces, we used the open balls to define the open sets. We can do a similar thing more generally.

**Def:** A basis $\mathcal{B} \subseteq \mathcal{P}(X)$ is a collection of subsets of $X$ satisfying the following properties:

1.) $X = \bigcup \mathcal{B}$

2.) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, $\exists$ $B_3 \in \mathcal{B}$ s.t.
$x \in B_3 \subseteq B_1 \cap B_2$




$\mathcal{B}$ is typically not a topology itself, but we can use it to construct one:

**Def:** The topology $\mathcal{T}$ generated by a basis $\mathcal{B}$ is defined as follows:

$$U \in \mathcal{T} \iff \forall\, x \in U \ \exists\, B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U.$$

**Claim:** $\mathcal{T}$ is in fact a topology.

**Pf:** Clearly $\emptyset, X \in \mathcal{T}$.

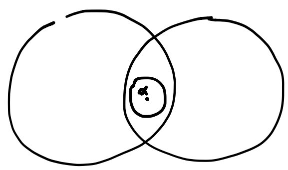
If $x \in \bigcup \mathcal{T}'$, then for some $U \in \mathcal{T}'$, $x \in B \subseteq U \subseteq \bigcup \mathcal{T}'$.

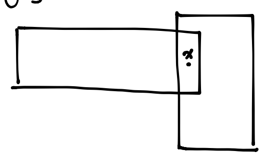
If $x \in \underbrace{U \cap V}_{\in \mathcal{T}}$, then $x \in B_1 \subseteq U$, $x \in B_2 \subseteq V$, so $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U \cap V$.

**Ex:** $\mathcal{B} = \{B_r(x) \mid r>0,\ x \in \mathbb{R}^n\}$ is a basis on $\mathbb{R}^n$, and it generates the standard topology.

and on a metric space in general

The set of rectangular regions is also a basis that generates the standard topology.





**Claim:** If $\mathcal{T}$ is the topology generated by basis $\mathcal{B}$, then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$ (including the empty union). (Exercise)

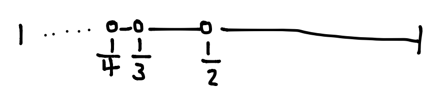
Note that open sets cannot be expressed uniquely as the union of basis elements.

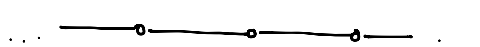
## Example: Open sets in $\mathbb{R}$

Consider the basis $\mathcal{B}$ consisting of open balls, i.e. open intervals.

If $U \subseteq \mathbb{R}$ is open, we may need infinitely many open sets to describe it.

E.g. $U = \bigcup_{n \in \mathbb{Z}} (n, n+1)$ or $U = \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n}\right)$



Sometimes we can't even order the basis elements though!

Ex: The complement of a cantor set:

$U_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$

$U_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$

$\vdots$


Question: Are any open sets in $\mathbb{R}$ the disjoint union of uncountably many open intervals? *

Ex: Other topologies on $\mathbb{R}$:

1.) $\mathbb{R}_\ell$ = $\mathbb{R}$ equipped w/ the lower limit topology $\mathcal{T}_\ell$,

which is generated by the basis $\{[a,b) \mid a < b\}$.

$[0,1)$ is not open in the standard topology, so $\mathcal{T}_\ell$ is not the standard topology.

However, for $x \in (a,b)$, $x \in [x,b) \subseteq (a,b)$, so $(a,b)$ is open in the standard topology $= \mathcal{T}$.

Thus $\mathcal{T} \subseteq \mathcal{T}_\ell$.

more open sets

fewer open sets

i.e. $\mathcal{T}_\ell$ is <u>finer</u> than $\mathcal{T}$, and $\mathcal{T}$ is <u>coarser</u> than $\mathcal{T}_\ell$.

**Lemma:** $\mathcal{B}$ and $\mathcal{B}'$ bases for $\mathcal{T}$ and $\mathcal{T}'$, respectively, on $X$.
Then $\mathcal{T}'$ is finer than $\mathcal{T} \iff \forall\, x \in X, B \in \mathcal{B}$ s.t. $x \in B$, $\exists\, B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

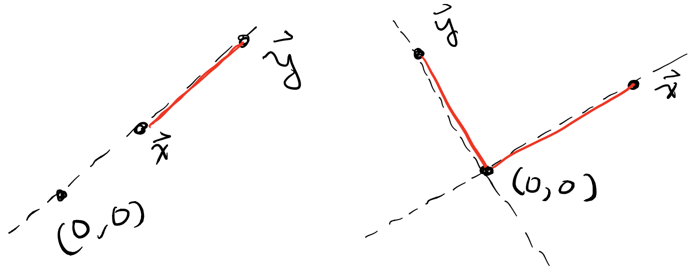
**Pf:** $\Leftarrow$: Let $U \in \mathcal{T}$. WTS $U \in \mathcal{T}'$. For $x \in U$, we can find $B \in \mathcal{B}$ s.t. $x \in B \subseteq U$. But then $\exists\; x \in B' \subseteq B \subseteq U$. so $U \in \mathcal{T}'$.

$\Rightarrow$: Let $x \in B$. Then $B \in \mathcal{T} \subseteq \mathcal{T}'$.
so $\exists\; B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$. $\square$

**Ex:** The discrete topology on $X$ has basis $\{\{x\} \mid x \in X\}$.
In fact $X$ has the discrete topology $\iff \{x\}$ is open $\forall\, x \in X$.

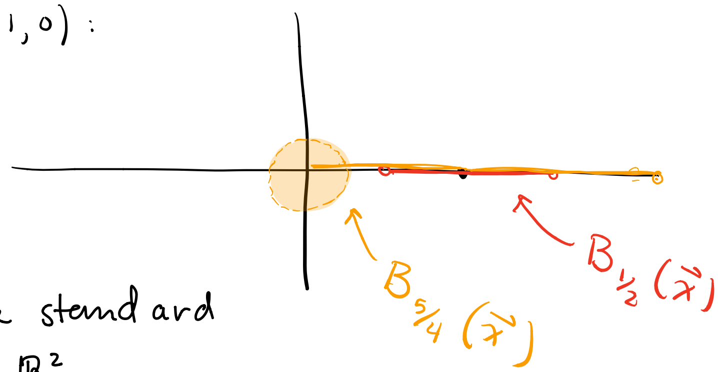
**Ex:** Consider the "Paris metric" on $\mathbb{R}^2$:

$$e(\vec{x}, \vec{y}) = \begin{cases} \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} & \text{if } \vec{x}, \vec{y} \text{ are on the same ray} \\ \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2} & \text{otherwise} \end{cases}$$




What do the open balls look like in this metric?

Around $\vec{x} = (1,0)$:




If $\mathcal{T}$ is the standard topology on $\mathbb{R}^2$, and $d$ the standard metric, then

$B_{e,1/2}(\vec{x}) \notin \mathcal{T}$, so $\mathcal{T}_p \not\subseteq \mathcal{T}$, if $\mathcal{T}_p$ is the topology induced by the Paris metric.

Is $\mathcal{T} \subseteq \mathcal{T}_p$?

Let $x \in \mathbb{R}^2$, and $r > 0$.

Since $d(\vec{x},\vec{y}) \leq e(\vec{x},\vec{y})$ by the triangle inequality, $\exists\ \varepsilon > 0$ s.t.

$$B_{e,\varepsilon}(\vec{y}) \subseteq B_{d,\varepsilon}(\vec{y}) \subseteq B_{d,r}(\vec{x}) \text{ for any } \vec{y} \in B_{d,r}(\vec{x})$$

So $\mathcal{T} \subseteq \mathcal{T}_p$ — the "Paris topology" is finer than the standard topology.

**Example: The subspace topology**

$X$ a topological space, $A \subseteq X$. If $X$ is a metric space, we already know how to put a metric, and thus a topology, on $A$

In general, the <u>subspace topology</u> on $A$ is defined

$$\mathcal{T}_A = \{ U \cap A \mid U \in \mathcal{T}_X \}$$

You can check using basic set operations that:

Claim:

1.) The subspace topology is a topology on A.

2.) If $\mathcal{B}$ is a basis for the topology on X, then

$$\mathcal{B}_A = \{B \cap A \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on A.

Ex: In $[0,\infty) \subseteq \mathbb{R}$ given the subspace topology,

$$(-1,1) \cap [0,\infty) = [0,1) \text{ is open in } [0,\infty) \text{ but not in } \mathbb{R}.$$

Example: The product topology

Let X and Y be topological spaces with topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Then the product topology on $X \times Y$ is generated by the basis

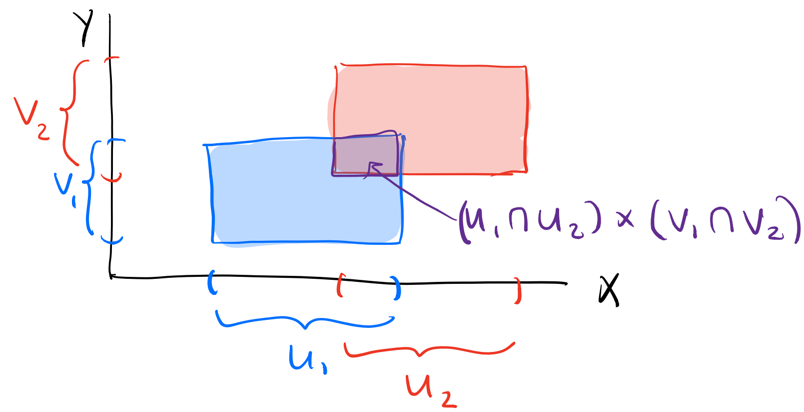
$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}, V \in \mathcal{T}'\}$$

Claim: $\mathcal{B}$ is a basis.

Pf: 1.) $X \times Y \in \mathcal{B}$, so the union certainly covers $X \times Y$.

2.) If $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$, then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = \underbrace{(U_1 \cap U_2)}_{\in \mathcal{T}} \times \underbrace{(V_1 \cap V_2)}_{\in \mathcal{T}'} \in \mathcal{B}. \quad \square$$

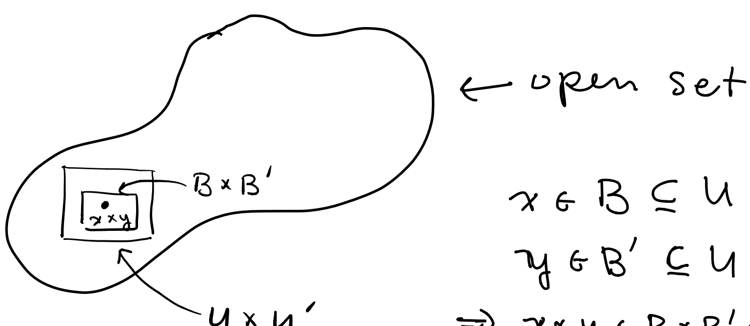



**Claim:** If $\mathcal{B}$ and $\mathcal{B}'$ are bases for $\mathcal{T}$ and $\mathcal{T}'$, respectively, then

$$\mathcal{D} = \{ B \times B' \mid B \in \mathcal{B},\ B' \in \mathcal{B}' \}$$

is a basis for the product topology.

**Pf sketch:**




$x \in B \subseteq U$

$y \in B' \subseteq U'$

$\Rightarrow x \times y \in B \times B' \subseteq U \times U'$

**Example: The order topology**

Let $X$ be a set with a total ordering i.e. for $a, b, c \in X$

- $a \leq b$ and $b \leq a \Rightarrow a = b$
- $a \leq b$ and $b \leq c \Rightarrow a \leq c$
- $a \leq b$ or $b \leq a$.

The order topology on $X$ is generated by open intervals

$(a,b)$ and possibly half open intervals $[a_0, b)$ and $(a, b_0]$

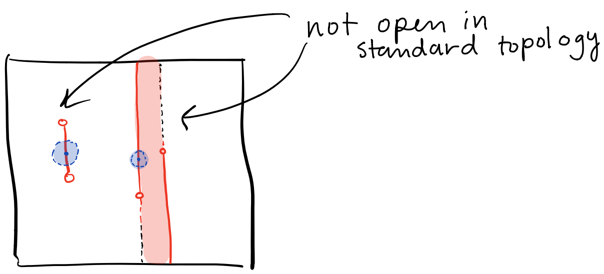
$a_0$: smallest element of $X$; $b_0$: largest element of $X$

if $X$ has a smallest and/or largest element.

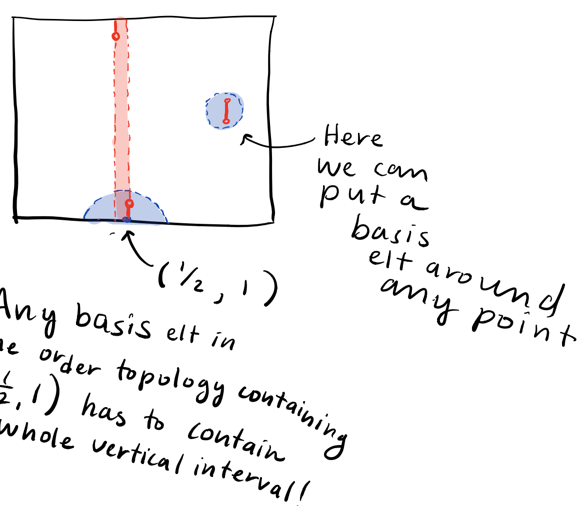
On $\mathbb{R}$, this generates the standard topology.

Ex: On $I \times I$, $I = [0,1]$, we can give the dictionary order where $a \times b < a' \times b'$ if $(a < a')$ or $(a = a'$ and $b < b')$.

Basis elements look like




Are the open sets in the standard topology open in the order topology?




So the two topologies are <u>not comparable</u> (i.e. neither is contained in the other).